Problem Session – STINT Meeting – January 2016

January 28, 2016

Sparse k-chromatic subgraph

Communicated by S. Thomassé by phone call.

Problem 1 (Thomassé)**.** Let k be a positive integer. Does there exists $f(k)$ and $g(k)$ such that if $\chi(G) \geq f(k)$, then G contains a subgraph H with $\chi(H) = k$ and $\Delta(H) \leq g(k)$?

The answer is trivial for $k = 2$ and easy for $k = 3$ because every graph with chromatic number at least 3 contains an odd cycle.

The same question replacing $\Delta(H)$ by $Ad(H)$ might be a first attempt to prove it.

Increasing path in edge-ordered graphs

Communicated by L. Esperet.

Let G be a graph and σ a numbering of the edges (every edge gets an integer distinct from the ones of the other edges). Let $p(G, \sigma)$ be the maximum length of an increasing path (the numbers increase along the path).

Let $\pi(G) = \min\{p(G, \sigma) \mid \sigma \text{ numbering of } E(G)\}$.

Problem 2 (Roditty, Shoam, and Yuster [7])**.** What is the maximum value of $\pi(G)$ when G is a planar graph ?

The value is at least 5 (see [7]) and at most 8 due to a result of Goncalves [6].

Question: what if we consider trails instead of paths ?

Decomposing a subcubic graph into a spanning, a matching and some cycles

Communicated by S. Pérennes.

Conjecture 3 (Hoffmann-Ostenhof)**.** Every connected subcubic graph can be decomposed into a spanning tree, a matching, and some cycle.

If there is a cycle whose removal does not disconnect the graph, then we can remove it, so we can remove the cycle.

N. Trotignon observed that Thomassen and Toft gave some properties of graph without disconnecting cycles. It might be interesting to have a look at it.

Proper orientation number

Communicated by F. Havet.

An orientation of a graph G is *proper* if for all $uv \in E(G)$, we have $d^-(u) \neq d^-(v)$. The *proper orientation number* of a graph G, denoted by $\mathrm{po}(G)$, is the minimum of the maximum indegree over all proper orientations of G. This number was introduced by Ahadi and Dehgan [2], who proved that $\chi(G) \leq \mathrm{po}(G) + 1 \leq \Delta(G) + 1$.

Many questions on the proper orientation number are open

Problem 4. Does there exist ϵ such that $\mathrm{po}(G) \leq \epsilon \cdot \chi(G) + (1-\epsilon)\Delta(G)$?

Problem 5. Does there exist a function f such that $\mathrm{po}(G) \leq f(\mathrm{tw}(G))$ where $tw(G)$ denotes the tree width of G ?

When $\mathrm{tw}(G) = 1$, i.e. when G is a forest, then Araujo et al. proved that $\mathrm{po}(G) \leq 4$. The question whether the proper orientation number of graphs with tree width 2 is widely open. It is not even known whether the proper orientation number of outerplanar graphs is bounded.

Clique- stable set separation

Communicated by A. Lagoutte.

A cut (A, B) *separates* two sets X and Y if $X \subseteq A$ and $B \subseteq B$.

A *clique-stable set separator*, or *CS-separator* for short, is a collections $\mathcal{C}$ of cuts such that for every disjoint clique K and stable set S there exists $(A, B) \in \mathcal{C}$ such that (A, B) separates K and S.

Answering a problem of Yannakakis, Göös showed that general graphs do not have a polynomial CS-separator.

Problem 6 (Maffray)**.** Does every perfect graph have a polynomial CS-separator ?

A subproblem which is still open is to consider *weakly chordal graphs*, which are graphs with no 5-hole and no 5-antihole.

Problem 7. Does every weakly chordal graph have a polynomial CS-separator ?

Diameter of the 7-colouring graph of a planar graph

Communicated by N. Bousquet.

Let G be a graph. Its k*-colouring graph* is the graph $C_k(G)$ whose vertices are the proper k-colourings of G, and two such colourings are adjacent in $C_k(G)$ if and only if they differ only in one vertex.

When G is planar, then the graph $C_7(G)$ is connected, while $C_6(G)$ may be disconnected. Moreover, Bousquet and Perarnau [5] proved that the diameter of $C_8(G)$ is bounded by a polynomial in $|G|$ for all planar graph G.

Problem 8. Let G be a planar graph. Is the diameter of $C_7(G)$ bounded by a polynomial in $|G|$?

Flows and bissection of cubic graphs in parts with small connected components

Communicated by L. Esperet.

Let r be a real such that $r \geq 1$. An r-flow of a graph is an orientation D of the arc of G together with a weighting w of the arcs in $[1, r-1]$ such that at each vertex v, $\sum_{u \in N^-(v)} w(uv) = \sum_{v \in N^+(v)} w(vw)$. The *real flow value* of G, denoted by $\Phi(G)$, is the minimum r such that G admits an r-flow.

Seymour 6-Flow Theorem implies that $\Phi(G) \leq 6$ for every bridgeless graph G and Tutte 5-Flow Conjecture states that $\Phi(G) \leq 5$ for every bridgeless graph. A first step to the conjecture would be to improve on 6.

Problem 9. Prove that $\Phi(G) < 6$ for every bridgeless graph G.

Consider a cubic graph and assume that it has an r-flow (D, w). It induces a bissection (V_1, V_2) where V_i is the set of vertices with out-degree i in D. Recall that a *bissection* of a graph is a bipartition in two part of the same size.

Moreover, one can check that every connected component in $G\langle V_i\rangle$ has size at most $r-2$. Therefore Seymour 6-Flow Theorem implies that every bridgeless cubic graph has a bissection (V_1, V_2) such that the connected components in $G\langle V_i\rangle$ have size at most 4, for $i = 1, 2$. It as been shown that every bridgeless cubic graph has a bissection (V_1, V_2) such that the connected components in $G\langle V_i\rangle$ have size at most 3, for $i = 1, 2$. The next step is the following conjecture.

Conjecture 10 (Ban and Linial [4])**.** Is it true that every bridgeless cubic graph G except the Petersen graph has a bissection (V_1, V_2) such that the connected components in $G\langle V_i\rangle$ have size at most 2, for $i = 1, 2$.

Remark 11. If we replace bisection by bipartition, the result is easy : consider a maximum cut (V_1, V_2). Every vertex has degree at most 1 in its part.

We cannot remove bridgeless in the above conjecture as shown by Esperet, Mazzioccolo and Tarsi. But there is another conjecture.

Conjecture 12 (Ban and Linial [4])**.** Is it true that every cubic graph G has a bipartition (V_1, V_2) such that $||V_1| - |V_2|| \leq 2$ and the connected components in $G\langle V_i\rangle$ have size at most 2, for $i = 1, 2$.

Optimal χ-bounding functions for graph with stability number at most 2.

Communicated by N. Trotignon.

Recall that $\alpha(G)$ is the maximum size of a stable set in a graph G.
One can easily show by induction on $\omega(G)$ that if $\alpha(G) \leq 2$, then $\chi(G) \leq \omega(G)(\omega(G)+1)/2$.

Problem 13. What is the smallest function f such that $\chi(G) \leq f(\omega(G))$ for all graph G with $\alpha(G) \leq 2$?

Using Ramsey theory, one can show that $f(\omega) = \Omega\left(\frac{\omega^2}{\log \omega}\right)$.

Detecting cycle with exactly two chords

Communicated by N. Trotignon.

It is well-known that is it polynomial-time solvable to decide whether a graph as a cycle with no chord as an induced subgraph (i.e. deciding whether a graph is not chrodal). Trotignon and Vušković [8] proved that it polynomial-time solvable to decide whether a graph as a cycle with exactly one chord as an induced subgraph.

Problem 14. Can we decide in polynomial time whether a graph contains a cycle with exactly two chords as an induced subgraph ?

Aboulker and Bousquet [1] proved that a graph with no such subgraph has chromatic number at most 6.

Subdivisions in digraphs with high chromatic number or high out-degree

Communicated by P. Aboulker.

Strong digraphs with high chromatic number

It is known that for every digraph H that is not a tree, there exists digraphs with arbitrarily high chromatic number that does not contain a subdivision of H as a subdigraph. The situation changes when we ask for the digraph to have high chromatic number and to be strongly connected.

Bondy proved that every strong digraph with chromatic number at least k contain a subdivision of the directed cycle of order k. Furthermore, Cohen, Havet, Lochet, and Nisse proved that for every cycle C with two blocks (The *blocks of a cycle* are its maximal subpaths.) or a the cycle with four blocks of length 1, there exists $f(C))$ such that every strong digraph with chromatic number at least $f(C)$ contains a subdivision of C. They conjecture the following.

Conjecture 15. For every oriented cycle C, there exists $f(C))$ such that every strong digraph with chromatic number at least $f(C)$ contains a subdivision of C.

More generally we may ask:

Question 16. For which digraph H there exists an integer $f(H)$ such that every strongly connected digraph D with $\chi(D) \geq f(H)$ contains a subdivision of H?

Digraphs with high minimimum out-degree

Conjecture 17 (Mader)**.** There exists an integer $f(k)$ such that every digraph D with $\delta^+(D) \geq f(k)$ contains a subdivision of the transitive tournament on k vertices.

$f(3) = 2$ is easy to prove and Mader proved that $f(4) = 3$, it is open for $k \geq 5$. Since a tournament on n vertices contains a transitive tournament on $\log n$ vertices, Mader's conjecture is equivalent to:

Conjecture 18 (Mader)**.** There exists an integer $f(k)$ such that every digraph D with $\delta^+(D) \geq f(k)$ contains a subdivision of a tournament on k vertices.

We can weaken Mader's Conjecture by asking the same question for acyclic digraphs. Aboulker, Cohen, Havet and Nisse proved that it holds for cycles with two blocks. Another interesting first step would be to prove it for oriented trees.

Conjecture 19. For every oriented tree T, there exists an integer $f(T)$ such that every digraph D with $\delta^+(D) \geq f(T)$ contains a subdivision of T.

References

[1] P. Aboulker and N. Bousquet. Excluding cycles with a fixed number of chords. *Discrete Applied Mathematics* 180:11-24, 2015.

[2] A. Ahadi and A. Dehgan. The complexity of the proper orientation number of regular graphs. Submitted to *Discrete Applied Mathematics*.

[3] J. Araujo, N. Cohen, S. F. de Rezende, F. Havet, and P. F Moura. On the proper orientation number of bipartite graphs. *Theoretical Computer Science 566*, 0 (2015), 59 – 75.

[4] A. Ban and N. Linial. Internal partitions of regular graphs. *Journal of Graph Theory*, to appear.

[5] N. Bousquet and G. Perarnau. Fast recoloring of sparse graphs. *European Journal of Combinatorics*, 52:1–11, 2016.

[6] D. Gonçalves. Caterpillar arboricity of planar graphs. *Discrete Math.* 307(16): 2112–2121, 2007.

[7] Y. Roditty, B. Shoham, and R. Yuster. Monotone paths in edge-ordered sparse graphs. . *Discrete Mathematics* 226:411–417, 2001.

[8] N. Trotignon, K. Vušković. A structure theorem for graphs with no cycle with a unique chord and its consequences. *J. Graph Theory* 63(1): 31-67, 2010.